

# Heat-kernels and functional determinants on the generalized cone

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*Abstract: We consider zeta functions and heat-kernel expansions on the bounded, generalized cone in arbitrary dimensions using an improved calculational technique. The specific case of a global monopole is analysed in detail and some restrictions thereby placed on the  $A_{5/2}$  coefficient. The computation of functional determinants is also addressed. General formulas are given and known results are incidentally, and rapidly, reproduced.*

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## 1 Introduction.

In this paper we refine and generalise techniques developed earlier for the evaluation of heat-kernel expansion coefficients and functional determinants of elliptic operators on manifolds with boundary. We concentrate on ball-like manifolds because precise answers can be found and, apart from illustrating our method, the results for such specific manifolds are often useful in restricting the general forms of heat-kernel coefficients.

One of the motivations for this paper is to compute for a particular curved manifold whose boundary is not geodesically embedded. The resulting restrictions are a little

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more informative than some others available [28, 29, 30]. The manifold also possesses a singularity, which increases its interest.

For calculational simplicity the operator is taken to be the modified Laplacian,  $\Delta - \xi R$ , acting on scalars. The analysis could be extended to forms without difficulty and also to other fields with a certain amount of extra work [23, 5, 24, 25, 26, 38, 31]. It is possible that our techniques will be of value in areas of physics where finite size systems and boundary effects play a role, such as quantum cosmology and statistical mechanics.

In the next section we outline the geometry we have in mind and discuss the eigenmodes. The  $\zeta$ -function is next constructed in section 3 and its properties translated into heat-kernel language in the following section. In order to make this paper reasonably self-contained the techniques alluded to previously are restated in improved and compactified form. The general method is applied to a global monopole in section 5 and the results used in section 7 to place restrictions on the numerical coefficients in a heat-kernel coefficient of some current mathematical interest [10]. Sections 9, 10 and 11 describe the evaluation of the functional determinant.

## 2 The geometry and eigenmodes.

The manifold in question can be termed the bounded, generalized cone and is defined as the  $(d + 1)$ -dimensional space  $\mathcal{M} = I \times \mathcal{N}$  with the hyperspherical metric *cf* [14]

$$ds^2 = dr^2 + r^2 d\Sigma^2, \quad (2.1)$$

where  $d\Sigma^2$  is the metric on the manifold  $\mathcal{N}$ , and  $r$  runs from 0 to 1.  $\mathcal{N}$  will be referred to as the base, or end, of the cone. If it has no boundary then it is the boundary of  $\mathcal{M}$ .

We note that the space is conformal to the product half-cylinder  $\mathbb{R}^+ \times \mathcal{N}$ ,

$$ds^2 = e^{-2x}(dx^2 + d\Sigma^2), \quad x = -\ln r, \quad (2.2)$$

which allows the curvatures on  $\mathcal{M}$  and  $\mathcal{N}$  to be related. The only nonzero components of the curvature on  $\mathcal{M}$  are, with obvious notation,

$$R^{ij}{}_{kl} = \frac{1}{r^2}(\hat{R}^{ij}{}_{kl} - (\delta_k^i \delta_l^j - \delta_l^i \delta_k^j)), \quad R_j^i = \frac{1}{r^2}(\hat{R}_j^i - (d-1)\delta_j^i), \quad R = \frac{1}{r^2}(\hat{R} - d(d-1)). \quad (2.3)$$

These measure the local deviation of  $\mathcal{N}$  from a unit  $d$ -sphere and indicate the existence of a singularity at the origin. Finally, the extrinsic curvature is  $\kappa_j^i = \delta_j^i$  and we recognise (2.3) at  $r = 1$  as the Gauss-Codacci equation.

Turning to analysis, the Laplacian is

$$\Delta_{\mathcal{M}} = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\mathcal{N}}. \quad (2.4)$$

Boundary conditions are imposed at  $r = 1$  as will be described in the next section.

The nonzero eigenmodes of  $\Delta_{\mathcal{M}}$  that are finite at the origin have eigenvalues  $-\alpha^2$  and are of the form

$$\frac{J_\nu(\alpha r)}{r^{(d-1)/2}} Y(\Omega), \quad (2.5)$$

where the harmonics on  $\mathcal{N}$  satisfy

$$\Delta_{\mathcal{N}} Y(\Omega) = -\lambda^2 Y(\Omega) \quad (2.6)$$

and

$$\nu^2 = \lambda^2 + (d-1)^2/4. \quad (2.7)$$

It is easiest to allow for the addition of the term  $-\xi R$  to  $\Delta_{\mathcal{M}}$  when  $\hat{R}$  is constant and we shall assume that this is so in the detailed calculations presented later in this paper. The obvious example is the sphere, discussed *passim* in section 5. If we are interested solely in the Laplacian ( $\xi = 0$ ) this restriction is unnecessary.

The modes will still be as in equation (2.5) with now

$$\nu^2 = \lambda^2 + (d-1)^2/4 + \xi(\hat{R} - d(d-1)) = \lambda^2 + \xi\hat{R} + d(d-1)(\xi_d - \xi) \quad (2.8)$$

where  $\xi_d = (d-1)/4d$ . For conformal coupling in  $d+1$  dimensions, the last term disappears, as it also does when  $d = 0$  or  $d = 1$ .

More generally, if  $\hat{R}$  is not constant, we write

$$\Delta_{\mathcal{M}} - \xi R = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{\xi d(d-1)}{r^2} + \frac{1}{r^2} (\Delta_{\mathcal{N}} - \xi \hat{R})$$

and introduce eigenfunctions,  $\bar{Y}$ , of the modified Laplacian on  $\mathcal{N}$ ,

$$(\Delta_{\mathcal{N}} - \xi \hat{R}) \bar{Y} = -\bar{\lambda}^2 \bar{Y},$$

so that the eigenfunctions on  $\mathcal{M}$  are again of the form (2.5) with  $Y$  replaced by  $\bar{Y}$  and

$$\nu^2 = \bar{\lambda}^2 + d(d-1)(\xi_d - \xi). \quad (2.9)$$

We assume that  $\nu \geq 1/2$  in order to avoid the appearance of types of solution other than (2.5).

### 3 The zeta function on $\mathcal{M}$ .

Let us first see how far the analysis can be taken without specifying the base manifold  $\mathcal{N}$ . A boundary value problem may still easily be posed due to the form of the chosen metric. Both Dirichlet and generalized Neumann (or Robin) boundary conditions are to be considered and in the notation of, for example, [30], these read explicitly

$$J_\nu(\alpha) = 0 \quad (3.1)$$

for Dirichlet and

$$\left(1 - \frac{D}{2} - \beta\right) J_\nu(\alpha) + \alpha J'_\nu(\alpha) = 0 \quad (3.2)$$

for Robin. We set  $D = d + 1$  and use  $D$  or  $d$ , whichever is convenient.

A handy way of organising eigenvalues is the Minakshisundaram-Pleijel  $\zeta$ -function. Let  $d(\nu)$  be the number of linearly independent scalar harmonics on  $\mathcal{N}$ . Then the base zeta function is defined by

$$\zeta_{\mathcal{N}}(s) = \sum d(\nu) \nu^{-2s} = \sum d(\nu) (\bar{\lambda}^2 + d(d-1)(\xi_d - \xi))^{-s} \quad (3.3)$$

and our first aim will be to express the whole zeta function on  $\mathcal{M}$ ,

$$\zeta_{\mathcal{M}}(s) = \sum \alpha^{-2s},$$

as far as possible in terms of this quantity. That is, we seek to replace analysis on the cone by that on its base in the manner of Cheeger for the infinite cone, [14].

We start with Dirichlet boundary conditions, the discussion for Robin conditions being virtually identical.

Following the analysis of [7, 8] the starting point is the representation of the zeta function in terms of a contour integral

$$\zeta_{\mathcal{M}}(s) = \sum d(\nu) \int_{\gamma} \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln J_\nu(k), \quad (3.4)$$

where the anticlockwise contour  $\gamma$  must enclose all the solutions of (3.1) on the positive real axis (for a similar treatment of the zeta function as a contour integral see [27, 6, 9]).

As was seen in [7, 8] it is very useful to split the zeta function into two parts. To actually perform this separation, some notation for the uniform asymptotic expansion of the Bessel function  $I_\nu(k)$  is required. For  $\nu \rightarrow \infty$  with  $z = k/\nu$  fixed one has, [35, 1],

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right], \quad (3.5)$$

with  $t = 1/\sqrt{1+z^2}$  and  $\eta = \sqrt{1+z^2} + \ln(z/(1+\sqrt{1+z^2}))$ . The first few coefficients are listed in [1]. Higher coefficients are immediately obtained by using the recursion [35, 1]

$$u_{k+1}(t) = \frac{1}{2}t^2(1-t^2)u'_k(t) + \frac{1}{8} \int_0^t d\tau (1-5\tau^2)u_k(\tau),$$

starting with  $u_0(t) = 1$ . As is clear, all the  $u_k(t)$  are polynomials in  $t$ . We also need the coefficients  $D_n(t)$  defined by the cumulant expansion

$$\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n} \quad (3.6)$$

which have the polynomial structure

$$D_n(t) = \sum_{b=0}^n x_{n,b} t^{n+2b}. \quad (3.7)$$

From the small  $z$  behaviour of eq.(3.5) one derives the important value  $D_n(1) = \zeta_R(-n)/n$ , which will be seen later on in eq. (9.2).

By adding and subtracting  $N$  leading terms of the asymptotic expansion, eq. (3.6), and performing the same steps as described in [7, 8] one finds the split

$$\zeta_{\mathcal{M}}(s) = Z(s) + \sum_{i=-1}^N A_i(s), \quad (3.8)$$

with the definitions

$$\begin{aligned} Z(s) = & \frac{\sin(\pi s)}{\pi} \sum_0^\infty \int dz (z\nu)^{-2s} \frac{\partial}{\partial z} \left( \ln \left( z^{-\nu} I_\nu(z\nu) \right) \right. \\ & \left. - \ln \left[ \frac{z^{-\nu}}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \right] - \sum_{n=1}^N \frac{D_n(t)}{\nu^n} \right), \end{aligned} \quad (3.9)$$

and

$$A_{-1}(s) = \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s+1)} \zeta_{\mathcal{N}}(s - 1/2), \quad (3.10)$$

$$A_0(s) = -\frac{1}{4} \zeta_{\mathcal{N}}(s), \quad (3.11)$$

$$A_i(s) = -\frac{1}{\Gamma(s)} \zeta_{\mathcal{N}}(s + i/2) \sum_{b=0}^i x_{i,b} \frac{\Gamma(s + b + i/2)}{\Gamma(b + i/2)}. \quad (3.12)$$

The function  $Z(s)$  is analytic on the strip  $(1 - N)/2 < \Re s$ , which may be seen by considering the asymptotics of the integrand in eq. (3.9) and by having in mind that the  $\nu^2$  are eigenvalues of a second order differential operator, see eq. (2.7).

As is clearly apparent in eq. (3.10)–(3.12), base contributions are separated from radial ones. We will see in the following section that this enables the heat-kernel coefficients of the Laplacian on the manifold  $\mathcal{M}$  to be written in terms of those on  $\mathcal{N}$ .

In order to treat Robin boundary conditions, only a few changes are necessary. In addition to expansion (3.5) we need [35, 1]

$$I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}(1+z^2)^{1/4}}{z} \left[ 1 + \sum_{k=1}^\infty \frac{v_k(t)}{\nu^k} \right], \quad (3.13)$$

with the  $v_k(t)$  determined by

$$v_k(t) = u_k(t) + t(t^2 - 1) \left[ \frac{1}{2} u_{k-1}(t) + t u'_{k-1}(t) \right]. \quad (3.14)$$

The relevant polynomials analogous to the  $D_n(t)$ , eq. (3.6), are defined by

$$\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k} + \frac{1 - D/2 - \beta}{\nu} t \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right) \right] \sim \sum_{n=1}^{\infty} \frac{M_n(t)}{\nu^n} \quad (3.15)$$

and have the same structure,

$$M_n(t) = \sum_{b=0}^n z_{n,b} t^{n+2b}. \quad (3.16)$$

One may again introduce a split as in eq. (3.8) with  $A_{-1}^R(s) = A_{-1}(s)$ ,  $A_0^R(s) = -A_0(s)$  and eq. (3.12) remains valid when  $x_{i,a}$  is replaced by  $z_{i,a}$ .

## 4 Heat-kernel coefficients on the generalized cone.

The previous formulas are already sufficient to give the heat-kernel coefficients on the manifold  $\mathcal{M}$  in terms of those on  $\mathcal{N}$ . However, before giving the relation, some special circumstances of our situation must be explained and, for expository purposes, a conventional short-time expansion of a generic heat-kernel will now be displayed,

$$K(t) \sim \sum_{n=0,1/2,1,\dots}^{\infty} C_n t^{n-D/2}. \quad (4.1)$$

For a geometric operator, such as the Laplacian on a Riemannian manifold, the coefficients  $C_n$  are integrals of polynomials in the curvatures and their derivatives. Typically, the volume integrand of  $C_n$  contains terms  $\sim R^n$  and it can be seen from the form of the curvature in the present case, (2.3), that the naively integrated coefficients  $C_n$  for  $n \geq D$  diverge. For this range, Cheeger [14] and Brüning [11] show that the relevant quantity in (4.1) is the *partie finie* of the integral. This is obtained, in our case, by restricting the radial integral to  $\epsilon \leq r \leq 1$ ,  $\epsilon > 0$ , and taking the finite remainder as  $\epsilon \rightarrow 0$ . An equivalent procedure for a given  $n$  is to evaluate in a dimension  $D > n$  and then continue to the required dimension, *à la* dimensional regularisation.

A further aspect of singular problems is the existence of logarithmic terms in the heat-kernel expansion, [12]. We introduce these via the important  $\zeta$ -function value  $\zeta(0)$  which is finite for a nonsingular elliptic operator on a smooth manifold. In the present case, (3.8) and (3.10) show that  $\zeta_{\mathcal{M}}(s)$  has a pole at  $s = 0$  provided  $\zeta_{\mathcal{N}}(s)$  has one at  $s = -1/2$ . According to a standard relation, the residue is proportional to the heat-kernel coefficient  $A_{(d+1)/2}^{\mathcal{N}}$  on  $\mathcal{N}$ , and so, if  $\mathcal{N}$  is closed, vanishes for even  $d$ , being then a pure boundary term. A pole at  $s = 0$  in the  $\zeta$ -function translates into a ‘nonstandard’ logarithm term in the heat-kernel expansion as we now show.

Let  $K_{\mathcal{M}}(t)$  be the heat-kernel associated with the modified Laplacian on  $\mathcal{M}$ , and define now the coefficients by the convention

$$K_{\mathcal{M}}(t) \sim \sum_{n=0,1/2,1,\dots}^{\infty} A_n^{\mathcal{M}} t^{n-D/2} + A' \log t. \quad (4.2)$$

A Mellin transform argument relating the heat-kernel to the  $\zeta$ -function, going back to [33], see also *eg* [39], gives

$$A_{n/2}^{\mathcal{M}} = \text{Res } \zeta_{\mathcal{M}}(s)\Gamma(s) \Big|_{s=(D-n)/2}, \quad n \neq D, \quad (4.3)$$

while

$$A_{D/2}^{\mathcal{M}} = \text{PP } \zeta_{\mathcal{M}}(0), \quad \text{and } A' = -\text{Res } \zeta_{\mathcal{M}}(0). \quad (4.4)$$

From (3.10)–(3.12) we find in particular

$$\begin{aligned} \text{PP } \zeta_{\mathcal{M}}(0) &= (\ln 2 - 1) \text{Res } \zeta_{\mathcal{N}}(-1/2) - \frac{1}{2} \text{PP } \zeta_{\mathcal{N}}(-1/2) - \frac{1}{4} \zeta_{\mathcal{N}}(0) \\ &\quad - \sum_{i=1}^{D-1} \frac{1}{i} \zeta_R(-i) \text{Res } \zeta_{\mathcal{N}}(i/2) \end{aligned} \quad (4.5)$$

and

$$\text{Res } \zeta_{\mathcal{M}}(0) = -\frac{1}{2} \text{Res } \zeta_{\mathcal{N}}(-1/2). \quad (4.6)$$

The singularity evidences itself in contributions to the constant and logarithmic terms in the expansion, *cf* [14].

In the later calculations it will be arranged that the logarithmic term does not occur. Its absence permits a standard evaluation of the functional determinant.

We consider an arbitrary dimension,  $D$ , and so, in practice, it will be enough to work with  $n < D$  in order to determine *any* coefficient. In consequence we use

$$A_{n/2}^{\mathcal{M}} = \Gamma((D-n)/2) \text{Res } \zeta_{\mathcal{M}}((D-n)/2), \quad (4.7)$$

in the following and continue in  $D$  as described above.

Denoting by  $A_n^{\mathcal{N}}$  the heat-kernel coefficients associated with  $\zeta_{\mathcal{N}}$  by the corresponding equations, we may write as an immediate consequence of eqs. (3.10)–(3.12) and (4.7) the basic relation,

$$\begin{aligned} A_{n/2}^{\mathcal{M}} &= \frac{1}{2\sqrt{\pi}(D-n)} A_{n/2}^{\mathcal{N}} - \frac{1}{4} A_{(n-1)/2}^{\mathcal{N}} \\ &\quad - \sum_{i=1}^{n-1} A_{(n-1-i)/2}^{\mathcal{N}} \sum_{b=0}^i x_{i,b} \frac{\Gamma((D-n+i)/2+b)}{\Gamma((D-n+i)/2) \Gamma(b+i/2)} \end{aligned} \quad (4.8)$$

with  $A_{(n-1)/2}^{\mathcal{N}} = 0$  for  $n = 0$ . Thus, given the coefficients on  $\mathcal{N}$ , eq. (4.8) relates them immediately to the coefficients on  $\mathcal{M}$ . The boundary condition at  $r = 1$  is encoded just in the sum over  $b$ . This relation can be used to restrict the general form of the heat-kernel coefficients as will be explained briefly in sections 6 and 7.

Eq. (4.8) remains true for Robin conditions once the sign of the second term on the right-hand side is reversed and the  $x$ 's replaced with the  $z$ 's.

## 5 The global monopole.

Let us illustrate this formalism with a bounded version of the simplified global monopole introduced by Sokolov and Starobinsky [36] and discussed more physically by Barriola and Vilenkin [4].

The manifold  $\mathcal{N}$  is a  $d$ -sphere of radius  $a$  and is the boundary  $\partial\mathcal{M}$  of  $\mathcal{M}$  at  $r = 1$ . If  $a$  is not unity, this produces a distortion which exhibits itself as a solid angle deficit, or excess, at the origin. The  $d + 1$ -space is not flat, unless  $a = 1$ , having scalar curvature

$$R = d(d-1)\frac{1-a^2}{a^2r^2}. \quad (5.1)$$

There are two useful conformal transformations. The first takes the metric into a product form (as in (2.2))

$$ds^2 = dr^2 + a^2r^2d\Omega^2 = e^{-2x}(dx^2 + a^2d\Omega^2), \quad x = -\ln r$$

which is that of the (Euclidean) Einstein universe,  $\mathbb{R}^+ \times S_a^d$ , where  $S_a^d$  is a sphere of radius  $a$ . This is conformally flat and so, therefore, is the monopole metric, a direct statement being

$$dr^2 + a^2r^2d\Omega^2 = \left(\frac{r}{l}\right)^{2\alpha}(dr^2 + r^2d\Omega^2)$$

where  $\alpha$  is given by ( $d \neq 0, 1$ )

$$\alpha = 1 - (1 + (1 - a^2)/a^2)^{1/2}$$

and  $l$  is an arbitrary scale parameter. This is reminiscent of a Schwarz-Christoffel transformation, the conformal nature of the map breaking down at the origin.

The first conformal relation gives the nonzero curvature components in terms of those on the unit sphere, (see (2.3)),

$$R^{ij}{}_{kl} = \frac{1-a^2}{a^2r^2}(\delta_k^i\delta_l^j - \delta_l^i\delta_k^j). \quad (5.2)$$

As explained earlier, the mode decomposition goes through exactly as in the flat case except that the order of the Bessel function acquires an extra shift,

$$\nu^2 = \frac{\lambda^2}{a^2} + \frac{(d-1)^2}{4} + \xi d(d-1)\frac{1-a^2}{a^2}$$

where  $\lambda^2$  are the eigenvalues of the Laplacian on the unit  $d$ -sphere. Hence (see (2.8))

$$\nu^2 = \frac{(n + (d-1)/2)^2}{a^2} + d(d-1)\frac{1-a^2}{a^2}(\xi - \xi_d)$$

so that if  $\xi = \xi_d$  we obtain the usual simplification (*cf* [32] for  $d = 2$ )

$$\nu = \frac{1}{a}\left(n + \frac{d-1}{2}\right), \quad n = 0, 1, 2, \dots,$$

and the base  $\zeta$ -function is given by a simple scaling of the unit sphere  $\zeta$ -function, which is the one appropriate for the uncompressed ball,

$$\zeta_{\mathcal{N}}(s) = a^{2s} \zeta_{S^d}(s). \quad (5.3)$$

A point to note is that there is no pole at  $s = 0$  in  $\zeta_{\mathcal{M}}(s)$ . As stated, this can only possibly occur if  $d$  is odd (for a closed  $\mathcal{N}$ ) and we know that the  $\zeta$ -function (5.3) is a finite sum of Riemann  $\zeta$ -functions and has no pole at  $s = -1/2$ . This is actually a consequence of our choice of conformal coupling.

In order to apply eq. (4.8) to the monopole, the residues of the base zeta function,  $\zeta_{\mathcal{N}}$ , are needed. These may be obtained most easily using its representation in terms of the Barnes zeta function [2] defined by the sum

$$\zeta_{\mathcal{B}}(s, a|\vec{r}) = \sum_{\vec{m}=0}^{\infty} \frac{1}{(a + \vec{m} \cdot \vec{r})^s}, \quad (5.4)$$

valid for  $\Re s > d$ , with the  $d$ -vectors  $\vec{m}$  and  $\vec{r}$ .

It is shown in [13] that the zeta function on a portion of the  $d$ -sphere determined by the degrees  $\vec{r}$  corresponds to the value  $a = \sum r_i - (d-1)/2$  for Dirichlet and to  $a = (d-1)/2$  for Neumann conditions on its perimeter. However we do not need the full generality of these statements and can limit ourselves to the case  $\vec{r} = \vec{1}$  corresponding to the hemisphere,

$$\zeta_{\mathcal{B}}(s, a|\vec{1}) = \sum_{l=0}^{\infty} \binom{l+d-1}{d-1} \frac{1}{(a+l)^s}. \quad (5.5)$$

It is easily shown, algebraically, that the full-sphere zeta function is the sum of the hemisphere Dirichlet and Neumann functions, [16]. To see this, remember that the number of independent scalar harmonics on  $\mathcal{N} = S^d$  is

$$d(l) = (2l+d-1) \frac{(l+d-2)!}{l!(d-1)!}. \quad (5.6)$$

For reasons of space, we do not give any history of sphere zeta functions.

Using the notation  $\zeta_{\mathcal{B}}(s, a|\vec{1}) = \zeta_{\mathcal{B}}(s, a)$  one finds

$$\zeta_{S^d}(s) = \zeta_{\mathcal{B}}(2s, (d+1)/2) + \zeta_{\mathcal{B}}(2s, (d-1)/2), \quad (5.7)$$

which reduces the analysis of the sphere zeta function to that of the Barnes function. Using the integral representation

$$\zeta_{\mathcal{B}}(s, a) = \frac{i\Gamma(1-s)}{2\pi} \int_L dz \frac{e^{z(d/2-a)} (-z)^{s-1}}{2^d \sinh^d(z/2)}, \quad (5.8)$$

where  $L$  is the Hankel contour, one immediately finds for the base function

$$\begin{aligned} \zeta_{\mathcal{N}}(s) &= a^{2s} \frac{i\Gamma(1-2s)}{2\pi} 2^{2s+1-d} \int_L dz (-z)^{2s-1} \frac{\cosh z}{\sinh^d z} \\ &= a^{2s} \frac{i\Gamma(2-2s)}{2\pi(d-1)} 2^{2s+1-d} \int_L dz (-z)^{2s-2} \frac{1}{\sinh^{d-1} z}. \end{aligned} \quad (5.9)$$

For the residues this yields ( $m = 1, 2, \dots, d$ )

$$\text{Res } \zeta_{\mathcal{N}}(m/2) = a^m \frac{2^{m-d} D_{d-m}^{(d-1)}}{(d-1)(m-2)!(d-m)!}, \quad (5.10)$$

with the  $D_{\nu}^{(d-1)}$  defined through (cf [15])

$$\left(\frac{z}{\sinh z}\right)^{d-1} = \sum_{\nu=0}^{\infty} D_{\nu}^{(d-1)} \frac{z^{\nu}}{\nu!}. \quad (5.11)$$

Obviously  $D_{\nu}^{(d-1)} = 0$  for  $\nu$  odd, so there are actually poles only for  $m = 1, 2, \dots, d$  with  $d - m$  even. The advantage of this approach is that known recursion formulas allow efficient evaluation of the  $D_{\nu}^{(n)}$  as polynomials in  $d$ , [34].

Using eq. (5.10) in eq. (4.8) we find for the heat-kernel coefficients  $A_{k/2}^{\mathcal{M}}$

$$\begin{aligned} \frac{(4\pi)^{D/2}}{|S^d|} A_{k/2}^{\mathcal{M}} &= \frac{(d-k-1)}{(d-1)(d-k+1)k!} \left(\frac{d+1-k}{2}\right)_{k/2} D_k^{(d-1)} a^{d-k} \\ &\quad - \frac{(d-k)}{4(d-1)(k-1)!} \left(\frac{d+2-k}{2}\right)_{(k-1)/2} D_{k-1}^{(d-1)} a^{d+1-k} \\ &\quad - \frac{2\sqrt{\pi}}{(d-1)} \sum_{i=1}^{k-1} \frac{d+i-k}{(k-1-i)!} \left(\frac{d+2-k+i}{2}\right)_{(k-i-1)/2} \times (5.12) \\ &\quad \sum_{b=0}^i \frac{x_{i,b}}{\Gamma(b+i/2)} \left(\frac{d+1-k+i}{2}\right)_b a^{d+1+i-k}, \end{aligned}$$

where  $(y)_n = \Gamma(y+n)/\Gamma(y)$  is the Pochhammer symbol. Eq.(5.12) exhibits the heat-kernel coefficients as explicit functions of the dimension  $d$  and, although derived for  $k < D$ , they can now be extended beyond this range.

For  $a = 1$  (5.12) reduces to the coefficients on the ball and is in full agreement with the results of Levitin [30]. The polynomials up to  $A_3^{\mathcal{M}}$  are listed in Appendix A.

For Robin boundary conditions one has to make the modifications outlined at the end of section 4. The results are summarized in Appendix B, once more up to  $A_3^{\mathcal{M}}$ .

## 6 Comparison with usual expressions.

The intention is to put restrictions on general forms of the  $A_{k/2}^{\mathcal{M}}$  using the particular results for the monopole. There is however the possible problem of a contribution from the singularity at the origin. Does a piece have to be added specially to the usual forms to account for this? The effect of the singularity appears only in the constant and logarithmic terms in the heat-kernel expansion and so only  $A_{D/2}^{\mathcal{M}}$  is affected. However the calculation provides unique polynomials in  $D$  for all  $k$ . Does anything special happen for  $k = D$ ? We show that it does and that singularity terms do not have to be added to the usual smooth forms. An example will illustrate the general point.

For  $D = 2$  the compressed ball is an ordinary cone of angle  $2\pi a$ . Consider now the usual Dirichlet smeared expression for a *smooth*  $D$ -manifold

$$(4\pi)^{D/2} A_1(f) = \frac{1}{6}(1 - 6\xi) \int_{\mathcal{M}} Rf + \frac{1}{3} \int_{\partial\mathcal{M}} (\kappa f - \frac{3}{2} n \cdot \partial f) \quad (6.1)$$

and, to avoid the log term, set  $\xi = \xi_d = (d-1)/4d$ . Substituting  $R$  from (5.1) and  $\kappa = d$ , (6.1) becomes, on the compressed ball,

$$A_1(f) = a^d \left( (3-d) \frac{1-a^2}{12a^2} \sum_{j=0}^{\infty} \frac{(d-1)f^{(2j)}(0)}{(d-1+2j)2j!} + \frac{d}{3}f(1) + \frac{1}{2}f'(1) \right) \frac{|S^d|}{(4\pi)^{D/2}}, \quad (6.2)$$

where we have assumed that the smearing function  $f$  depends on  $r^2$  only.

Note that the  $(d-1)$  factor, making  $R$  vanish on the disc, cancels against a corresponding factor from the integration over  $r$  for  $j=0$  so that the volume term remains nonzero at  $D=2$ . Then, evaluating at  $D=2$  gives

$$A_1(f) = \frac{1}{12} \left( \frac{1-a^2}{a} f(0) + 2af(1) \right) + \frac{a}{2} f'(1) \quad (6.3)$$

which can be compared with the standard expression for a 2-manifold with a conical singularity of angle  $\beta$  at the origin,

$$A_1(f) = \frac{1}{24\pi} (1 - 6\xi) \int_{\mathcal{M}} Rf + \frac{1}{12\pi} \int_{\partial\mathcal{M}} (\kappa f - \frac{3}{2} n \cdot \partial f) + \frac{1}{12} \left( \frac{2\pi}{\beta} - \frac{\beta}{2\pi} \right) f(0), \quad (6.4)$$

usually derived from the Sommerfeld-Carslaw heat-kernel on the cone. (It can be generalised to any dimension.)

Evaluated directly on the compressed 2-ball where  $R=0$  now and  $\kappa=1$ , (6.4) agrees with (6.3). We see that the singularity part of (6.4) arises as the  $D \rightarrow 2$  limit of the volume integration over the monopole curvature densities in the usual *smooth* expression. In this way the detailed analysis of the cone heat-kernel could be avoided. This is also true if  $\xi \neq \xi_d$  because of the  $(d-1)$  factor in the eigenvalues, eqs. (2.8) or (2.9).

The  $D=4$  case can be investigated in a similar fashion by examining  $A_2$ . In the general case, a value being fixed for  $k$ , the volume integrand of  $A_{k/2}$  (if there is one) vanishes at  $D=k$  because of the conformal flatness. However the *integrated* volume  $A_{k/2}$  remains nonzero and is the contribution of the singularity.

Another application of the  $A_2$  coefficient presents itself. For the ordinary cone, it is known that the smeared heat-kernel expansion consists of a series of rational functions in the apex angle which are straightforwardly calculated as residues from the Sommerfeld-Carslaw expression. The singularity term in (6.4) is the first of these functions. The second will come from the smeared  $A_2$  evaluated at  $D=2$ . If  $f(r^2)$  is Taylor expanded about the origin, because of the factor  $(d-1)$  in (5.1) all terms in  $f$  in the volume part will vanish except that proportional to  $r^2$  which yields a factor of  $(d-1)$  in the denominator giving a nonzero result. This would allow one to obtain the second of these residue functions, although this is not the best way. The upshot is that the conical singularity in

two dimensions can be exactly simulated by a monopole in  $D$  dimensions as the  $D \rightarrow 2$  limit of the smooth formulation.

The main conclusion of this section is that the polynomial forms deduced in the present paper for the monopole can be compared immediately with the usual smooth general forms, in so far as these are known. This we proceed to do in a particular case.

## 7 The $A_{5/2}$ coefficient.

Branson, Gilkey and Vassilevich [10] (Lemma 5.1) give the general form of this coefficient and determine many of the numerical coefficients. Looking at the  $A_{5/2}$  expression for the global monopole, section 5 and Appendices A and B, we are able to fix some additional numbers. We do not give here a comprehensive treatment of this question, intending only that it should illustrate our general method. It is possible that the restrictions could be found by easier means.

Using Lemma 5.1 of ref. [10] for the global monopole, inserting the geometric tensors given in the previous sections and comparing with the polynomials in Appendices A and B, we find for Dirichlet boundary conditions,

$$d_{36}^- = -\frac{65}{128}; \quad d_{37}^- = -\frac{141}{32}; \quad d_{40}^- = -\frac{327}{8},$$

together with the relations,

$$\begin{aligned} d_{38}^- + d_{39}^- &= 1049/32, \\ d_1^- + 2d_{27}^- - 2d_{29}^- &= -504, \\ d_1^- - 4d_2^- - 2d_{25}^- &= -360. \end{aligned} \tag{7.1}$$

For Robin boundary conditions the results read

$$\begin{aligned} d_{21}^+ &= -60; & d_{30}^+ &= 2160; & d_{31}^+ &= 1080; \\ d_{32}^+ &= 360; & d_{33}^+ &= 885/4; & d_{34}^+ &= 315/2; \\ d_{35}^+ &= 150; & d_{36}^+ &= 2041/128; & d_{37}^+ &= 417/32; \\ d_{40}^+ &= 231/8, \end{aligned} \tag{7.2}$$

with the additional relations,

$$\begin{aligned} d_{38}^+ + d_{39}^+ &= 1175/32, \\ d_1^+ + 2d_{27}^+ - 2d_{29}^+ &= 186, \\ d_1^+ - 4d_2^+ - 2d_{25}^+ &= -130. \end{aligned} \tag{7.3}$$

For Dirichlet conditions our example thus reduces the number of unknown numerical coefficients effectively by 6, and for Robin by 13.

Eq. (4.8) also allows one to place restrictions on the general form of the coefficients which we want to describe briefly. Assume that  $\mathcal{N}$  is closed and thus has no boundary so

$\partial\mathcal{M} = \mathcal{N}$  and  $A_{n/2}^{\mathcal{N}} = 0$  for  $n$  odd. The idea will already be clear from the lowest example  $n = 1$ . Then eq. (4.8) gives

$$A_{1/2}^{\mathcal{M}} = -\frac{1}{4}A_0^{\mathcal{N}} = -\frac{1}{4}(4\pi)^{(D-1)/2} |\partial\mathcal{M}|,$$

which is of course the known result. As a rule, knowing the volume coefficient  $A_n^{\mathcal{N}}$ , relation (4.8) puts restrictions on the coefficient  $A_{n+1/2}^{\mathcal{M}}$ . Let us illustrate this a bit more. Choosing the operator  $-(\Delta + E)$ ,  $E$  depending only on the coordinates on  $\mathcal{N}$ ,  $A_{3/2}^{\mathcal{M}}$  has the structure (we use no smearing function this time)

$$A_{3/2}^{\mathcal{M}} = -(384)^{-1}(4\pi)^{(D-1)/2} \int_{\partial\mathcal{M}} (d_1 E + d_2 R + d_3 R_{ab} n^a n^b + d_4 \kappa^2 + d_5 \kappa_{ab} \kappa^{ab}).$$

Employing eq. (4.8) again,

$$A_{3/2}^{\mathcal{M}} = -\frac{1}{4}A_1^{\mathcal{N}} - \frac{1}{64}A_0^{\mathcal{N}}(11 - 12D + 5D^2),$$

and also the Gauss-Codacci relations between the intrinsic and ambient geometries of  $\mathcal{N}$ , one finds the known numbers,

$$d_1 = 96; \quad d_2 = 16; \quad d_4 = 7; \quad d_5 = -10,$$

just from a knowledge of the volume term  $A_1^{\mathcal{N}}$ . Several numerical coefficients might also be determined for the  $A_{5/2}^{\mathcal{M}}$  coefficient in this way but, taking the computations of ref. [10] and the previous results together, probably nothing new would emerge.

## 8 Lens space bases.

Examples of locally spherical bases,  $\mathcal{N}$ , are the homogeneous spaces  $S^d/\Gamma$  where  $\Gamma$  is a discrete group of free isometries of  $S^d$ . The corresponding infinite cone ( $0 \leq r < \infty$ ) has been treated in [17]. For simplicity the sphere radius is set to unity to make the cones locally flat and we consider only the value  $\zeta_{\mathcal{M}}(0)$  in detail. Because of the homogeneity, the heat-kernel coefficients on  $S^d/\Gamma$  are simply a factor of  $|\Gamma|$  smaller than those on the unidentified sphere and, therefore, so are those on  $\mathcal{M}$  computed according to (4.8).

For even  $D$ , equation (4.5) reduces to

$$\zeta_{\mathcal{M}}(0) = -\frac{1}{2}\zeta_{\mathcal{N}}(-1/2) - \sum_{i=1}^d \frac{1}{i} \zeta_R(-i) \text{Res } \zeta_{\mathcal{N}}(i/2) \quad (8.1)$$

the first part of which we recognise as being the negative of the total Casimir energy on  $\mathcal{N}$ . The second part is written by homogeneity in terms of the full sphere value

$$\sum_{i=1}^d \frac{1}{i} \zeta_R(-i) \text{Res } \zeta_{S^d/\Gamma}(i/2) = \frac{1}{|\Gamma|} \sum_{i=1}^d \frac{1}{i} \zeta_R(-i) \text{Res } \zeta_{S^d}(i/2). \quad (8.2)$$

To obtain the contribution due solely to the singularity,  $1/|\Gamma|$  of the complete sphere value must be subtracted from (8.1). We see that the last term cancels and so

$$\zeta_{\mathcal{M}}^{\text{sing}}(0) = -\frac{1}{2} \left( \zeta_{S^d/\Gamma}(-1/2) - \frac{1}{|\Gamma|} \zeta_{S^d}(-1/2) \right). \quad (8.3)$$

As a simple case consider the three-dimensional lens space ( $\Gamma = \mathbb{Z}_m$ ). Then

$$\zeta_{\mathcal{M}}^{\text{sing}}(0) = \frac{1}{720m} (m^2 - 1)(m + 11) \quad (8.4)$$

using the Casimir energy calculated in [18]. This agrees of course with the value in [17]. The higher-dimensional cases, and other groups, can be treated by various means.

## 9 Dirichlet functional determinants.

As a further application of the ideas presented in the previous sections, let us consider the functional determinant on the generalized cone. To avoid problems of definition, we must assume that  $A_{(d+1)/2}^{\mathcal{N}}$  vanishes. This eliminates the possibility of a pole in  $\zeta_{\mathcal{M}}(s)$  at  $s = 0$  and the determinant is then conventionally defined to be  $\exp(-\zeta'_{\mathcal{M}}(0))$ .

For the calculation of the determinant we have seen in [8] that the first  $D - 1 = d$  terms in the asymptotic expansion are to be removed. Thus, from now on, we set  $N = d$  in eq. (3.8). The contribution of the  $A_i$ 's to the determinant may be immediately given,

$$\begin{aligned} A'_{-1}(0) &= (\ln 2 - 1) \zeta_{\mathcal{N}}(-1/2) - \frac{1}{2} \zeta'_{\mathcal{N}}(-1/2), \\ A'_0(0) &= -\frac{1}{4} \zeta'_{\mathcal{N}}(0), \\ A'_i(0) &= -\frac{\zeta_R(-i)}{i} \left( \gamma \text{Res } \zeta_{\mathcal{N}}(i/2) + \text{PP } \zeta_{\mathcal{N}}(i/2) \right) \\ &\quad - \sum_{b=0}^i x_{i,b} \psi(b + i/2) \text{Res } \zeta_{\mathcal{N}}(i/2), \end{aligned} \quad (9.1)$$

with  $\psi(x) = (d/dx) \ln \Gamma(x)$ . Following the procedure in [8] we find

$$Z'(0) = \sum d(\nu) \int_0^\infty dt \left( \sum_{n=1}^d \frac{D_n(1)}{(n-1)!} t^n + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-t\nu}}{t}, \quad (9.2)$$

which is well defined by construction, as is seen explicitly using the small  $t$  asymptotic expansion,

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} - \sum_{n=1}^\infty \frac{t^n}{n!} \zeta_R(-n), \quad (9.3)$$

and the value  $D_n(1) = \zeta_R(-n)/n$ . This expansion may also be used to obtain a kind of asymptotic series for  $Z'(0)$ ,

$$Z'(0) = \sum_{n=d+1}^{\infty} \frac{\zeta_R(-n)}{n} \zeta_{\mathcal{N}}(n).$$

However, as a rule,  $\zeta_{\mathcal{N}}(n)$  can be determined only numerically once the eigenvalues are known.

Introducing the ‘square root’ heat-kernel associated with the eigenvalue  $\nu$ ,

$$K_{\mathcal{N}}^{1/2}(t) = \sum d(\nu) e^{-t\nu},$$

eq. (9.2) can be written in the form

$$Z'(0) = \int_0^{\infty} dt \frac{1}{t} \left( \sum_{n=1}^d \frac{D_n(1)}{(n-1)!} t^n + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) K_{\mathcal{N}}^{1/2}(t).$$

Let us calculate the individual pieces, as far as possible. For this purpose, introduce a regularisation parameter,  $z$ , and define

$$Z'(0, z) = \int_0^{\infty} dt t^{z-1} \left( \sum_{n=1}^d \frac{D_n(1)}{(n-1)!} t^n + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) K_{\mathcal{N}}^{1/2}(t).$$

This leads to

$$\begin{aligned} Z'(0, z) = & \sum_{n=1}^d \frac{D_n(1)}{(n-1)!} \Gamma(n+z) \zeta_{\mathcal{N}}\left(\frac{z+n}{2}\right) + \frac{1}{2} \zeta_{\mathcal{N}}\left(\frac{z}{2}\right) \Gamma(z) \\ & - \zeta_{\mathcal{N}}\left(\frac{z-1}{2}\right) \Gamma(z-1) + \zeta_{\mathcal{N}+1}(z) \Gamma(z), \end{aligned} \quad (9.4)$$

which we need at  $z = 0$ . Here we have introduced, as seems natural, the zeta function

$$\zeta_{\mathcal{N}+1}(z) = \sum_{n=1}^{\infty} \sum d(\nu) (\nu + n)^{-z} = \frac{1}{\Gamma(z)} \sum d(\nu) \int_0^{\infty} dt t^{z-1} \frac{e^{-t\nu}}{e^t - 1}. \quad (9.5)$$

Eq. (9.4) may be expanded around  $z = 0$  and the required derivative expressed in the intermediate form

$$\begin{aligned} \zeta'_{\mathcal{M}}(0) = & \sum_{i=1}^d \text{Res } \zeta_{\mathcal{N}}(i/2) \left( \frac{\zeta_R(-i)}{i} (-\gamma + 2\psi(i)) - \sum_{b=0}^i x_{i,b} \psi(b + i/2) \right) \\ & - \frac{1}{2} \gamma \zeta_{\mathcal{N}}(0) + (\ln 2 - \gamma) \zeta_{\mathcal{N}}(-1/2) \\ & + \lim_{z \rightarrow 0} \left( \sum_{i=1}^d \frac{2}{zi} \zeta_R(-i) \text{Res } \zeta_{\mathcal{N}}(i/2) + \frac{1}{2z} \zeta_{\mathcal{N}}(0) + \frac{1}{z} \zeta_{\mathcal{N}}(-1/2) + \Gamma(z) \zeta_{\mathcal{N}+1}(z) \right). \end{aligned} \quad (9.6)$$

Several nonlocal pieces, difficult to determine, have cancelled between  $Z'(0)$  and the  $A_i(s)$ .

The small  $z$  expansion,

$$\Gamma(z)\zeta_{\mathcal{N}+1}(z) \sim \frac{1}{z}\zeta_{\mathcal{N}+1}(0) - \gamma\zeta_{\mathcal{N}+1}(0) + \zeta'_{\mathcal{N}+1}(0),$$

must now be employed where the value of  $\zeta_{\mathcal{N}+1}(0)$  follows from the fact, [39], that it equals the  $C_D$  term in the asymptotic  $t \rightarrow 0$  expansion of

$$\sum d(\nu) \frac{e^{-t\nu}}{e^t - 1} = \sum_{n=0}^{\infty} C_n t^{n-D},$$

and can be found using (9.3). Explicitly

$$\zeta_{\mathcal{N}+1}(0) = -\frac{1}{2}\zeta_{\mathcal{N}}(0) - \zeta_{\mathcal{N}}(-1/2) - 2 \sum_{i=1}^d \text{Res } \zeta_{\mathcal{N}}(i/2) \frac{\zeta_R(-i)}{i}. \quad (9.7)$$

Using the above results and notation, the derivative emerges in the final form

$$\begin{aligned} \zeta'_{\mathcal{M}}(0) &= \zeta'_{\mathcal{N}+1}(0) + \ln 2 \left( \zeta_{\mathcal{N}}(-1/2) + 2 \sum_{i=1}^d \text{Res } \zeta_{\mathcal{N}}(i/2) D_i(1) \right) \\ &\quad + 2 \sum_{i=1}^d \text{Res } \zeta_{\mathcal{N}}(i/2) \left( D_i(1) \sum_{k=1}^{i-1} \frac{1}{k} + \int_0^1 \frac{D_i(t) - tD_i(1)}{t(1-t^2)} dt \right) \end{aligned} \quad (9.8)$$

after some minor manipulation. It is seen that all  $\gamma$ -dependent pieces have cancelled and, in short, apart from contributions coming from the  $\zeta_{\mathcal{N}+1}(z)$  piece, the functional determinant is determined through the leading heat-kernel coefficients on the manifold  $\mathcal{N}$ .

It does not seem possible to proceed much further for the general case because there is no explicit expression for  $\zeta_{\mathcal{N}+1}(z)$ . The best we have found is the integral representation

$$\zeta_{\mathcal{N}+1}(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)} \zeta_R(z-s) \zeta_{\mathcal{N}}(s/2), \quad (9.9)$$

with  $\Re c > d$ , which one may find starting from (9.5) using the Mellin Barnes integral representation of the exponential function. Equation (9.7) is recovered in this representation closing the contour to the left.

However, for the example of the monopole, one can continue directly, as will be shown in the next section.

## 10 Monopole determinant.

A situation of possible physical significance is the global monopole. In the infinite case, Mazzitelli and Lousto [32] have evaluated some local vacuum averages on  $\mathbb{R} \times \mathcal{M}$ . In the bounded case we can find the effective one-loop action on  $\mathcal{M}$  in the guise of the functional determinant and so we now specialise  $\mathcal{M}$  to be the global monopole of section 5.

For conformal coupling we had

$$\nu = \frac{1}{a} \left( l + \frac{d-1}{2} \right)$$

and the base zeta function, eq.(5.3), is

$$\zeta_{\mathcal{N}}(s) = a^{2s} \sum_{l=0}^{\infty} d(l) \left( l + \frac{d-1}{2} \right)^{-2s}.$$

Furthermore we have

$$\begin{aligned} \zeta_{\mathcal{N}+1}(s) &= a^s \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} d(l) \left( l + an + \frac{d-1}{2} \right)^{-s} \\ &= a^s \left( \zeta_{\mathcal{B}}(s, (d+1)/2 + a|\vec{r}) + \zeta_{\mathcal{B}}(s, (d-1)/2 + a|\vec{r}) \right), \end{aligned} \quad (10.1)$$

with  $\vec{r} = (\vec{1}, a)$ ,  $\vec{1}$  being the  $d$ -dimensional unit vector. Thus, together with eq. (9.8), the functional determinant has been found in terms of derivatives of Barnes zeta functions and a given polynomial in the radius  $a$  and the dimension  $d$ . The polynomial follows from eqs. (5.10) and (5.9),

$$\begin{aligned} \zeta_{\mathcal{N}}(0) &= -\frac{2^{1-d}}{(d-1)d!} D_d^{(d-1)}, \\ \zeta_{\mathcal{N}}(-1/2) &= \frac{2^{1-d}}{a(d-1)(d+1)!} D_{d+1}^{(d-1)}. \end{aligned}$$

For arbitrary radius  $a$  it seems that one cannot construct the analytical continuation needed to find an explicit expression for  $\zeta'_{\mathcal{N}+1}(0)$ . A numerical treatment could start immediately from eqs. (9.1) and (9.2) or from the formulas in section 39 of ref. [2]. For a rational radius one can go further, as explained for example in [19], however we proceed here only for the ball, *i.e.*  $a = 1$ . Then, the zeta function, eq. (10.1), may be expressed in terms of just Hurwitz zeta functions in the following way.

First one finds

$$\zeta_{\mathcal{N}+1}(s) = \sum_{l=0}^{\infty} e(l) \left( l + \frac{d+1}{2} \right)^{-s}$$

with the “degeneracy”

$$e(l) = (2l+d) \frac{(l+d-1)!}{l! d!}.$$

Then, expanding  $e(l)$  in standard fashion as

$$e(l) = \sum_{\alpha=0}^d e_{\alpha} \left( l + \frac{d+1}{2} \right)^{\alpha},$$

the representation

$$\zeta_{\mathcal{N}+1}(s) = \sum_{\alpha=0}^d e_{\alpha} \zeta_H(s - \alpha; (d+1)/2) \quad (10.2)$$

in terms of the Hurwitz zeta function,  $\zeta_H$ , follows. So finally (*cf* [3] p432)

$$\zeta'_{\mathcal{N}+1}(0) = \sum_{\alpha=0}^d e_{\alpha} \zeta'_H(-\alpha; (d+1)/2). \quad (10.3)$$

All quantities needed to calculate the functional determinant on the ball are now provided. The results agree with previous ones presented in [8, 19, 20, 21] for dimensions  $D$  from 2 to 8. The structure of those results, such as the absence of the constant  $\gamma$ , the appearance of the derivatives of Riemann zeta functions with arguments up to  $-d$  and a certain prefactor of the  $\ln 2$  term, is made completely clear with eqs. (9.8) and (10.3) and is now shown to be true for all dimensions  $D$ .

## 11 Robin functional determinants.

Let us describe briefly the analogous treatment for Robin boundary conditions. Having the comments at the end of section 3 in mind, the contributions coming from the  $A_i$ 's are given by eq. (9.1) with the mentioned replacements. Following once more the lines of [8] we find

$$Z'_R(0) = Z'(0) + N(u), \quad (11.1)$$

with  $N(u)$  given by

$$N(u) = \sum d(\nu) \left( -\ln \left( 1 + \frac{u}{\nu} \right) + \sum_{n=1}^d (-1)^{n+1} \frac{1}{n} \left( \frac{u}{\nu} \right)^n \right),$$

and  $u = 1 - D/2 - \beta$ . Thus for Robin conditions we have to treat only one additional piece, the last one in eq. (11.1), in order to reach the result analogous to eq. (9.8) for Dirichlet conditions.

To proceed, write  $N(u)$  in the form

$$N(u) = \sum d(\nu) \int_0^{\infty} dt \frac{e^{-\nu t}}{t} \left( e^{-ut} + \sum_{n=0}^d (-1)^{n+1} \frac{u^n t^n}{n!} \right),$$

and again introduce a regularization parameter  $z$ , as in the derivation of eq. (9.4). We find for the resulting function,  $N(u, z)$ ,

$$N(u, z) = \zeta_{\mathcal{N}}(z, u) \Gamma(z) + \sum_{n=0}^d (-1)^{n+1} \frac{u^n}{n!} \Gamma(z+n) \zeta_{\mathcal{N}}((z+n)/2),$$

where we have introduced

$$\zeta_{\mathcal{N}}(z, u) = \frac{1}{\Gamma(s)} \sum d(\nu) \int_0^\infty dt t^{z-1} e^{-(\nu+u)t}.$$

One easily obtains as explained previously

$$\zeta_{\mathcal{N}}(0, u) = \zeta_{\mathcal{N}}(0) + 2 \sum_{l=1}^d (-1)^l \frac{u^l}{l} \text{Res } \zeta_{\mathcal{N}}(l/2),$$

which guarantees that the limit  $z \rightarrow 0$  is well defined. These relations lead to

$$\begin{aligned} N(u) &= \zeta'_{\mathcal{N}}(0, u) - \frac{1}{2} \zeta'_{\mathcal{N}}(0) \\ &+ \sum_{n=1}^d (-1)^{n+1} \frac{u^n}{n} \left( 2 \text{Res } \zeta_{\mathcal{N}}(n/2) (\psi(n) + \gamma) + \text{PP } \zeta_{\mathcal{N}}(n/2) \right). \end{aligned}$$

As in Dirichlet conditions, on adding up all contributions to the required derivative, several pieces cancel leaving the final compact form

$$\begin{aligned} \zeta_{\mathcal{M}}^{R'}(0) &= \zeta'_{\mathcal{N}+1}(0) + \zeta'_{\mathcal{N}}(0, u) + \ln 2 \left( \zeta_{\mathcal{N}}(-1/2) + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^d \text{Res } \zeta_{\mathcal{N}}(i/2) M_i(1) \right) \\ &+ 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^d \text{Res } \zeta_{\mathcal{N}}(i/2) \left( M_i(1) \sum_{k=1}^{i-1} \frac{1}{k} + \int_0^1 \frac{M_i(t) - t M_i(1)}{t(1-t^2)} dt \right) \\ &+ 2 \sum_{\substack{i=1 \\ i \text{ even}}}^d \text{Res } \zeta_{\mathcal{N}}(i/2) \left( M_i(1) \sum_{k=1}^{i-1} \frac{1}{k} + \int_0^1 \frac{M_i(t) - t^2 M_i(1)}{t(1-t^2)} dt \right) \end{aligned} \quad (11.2)$$

where  $M_i(1) = D_i(1) + (-1)^{i+1} u^i / i$ .

This completely parallels eq. (9.8) for Dirichlet conditions. Again, the  $\gamma$ -dependence has disappeared, and the nonlocal pieces are clearly confined to the first two terms which have to be seen as special functions as they stand. Nothing more can be said without specializing to simple manifolds.

Let us briefly describe the simplifications occurring for the monopole. All pieces are known from the Dirichlet case apart from

$$\begin{aligned} \zeta_{\mathcal{N}}(s, u) &= a^s \left( \zeta_{\mathcal{B}}(s, (d+1)/2 + au) + \zeta_{\mathcal{B}}(s, (d-1)/2 + au) \right) \\ &= a^s \sum_{l=0}^{\infty} d(l) (l + (d-1)/2 + au)^{-s}. \end{aligned}$$

However, using the procedure explained at the end of section 10, we expand

$$d(l) = \sum_{\alpha=0}^{d-1} e_{\alpha}(au) (l + (d-1)/2 + au)^{\alpha},$$

and then, since the  $e_\alpha(au)$  are polynomials in  $au$ ,  $\zeta_{\mathcal{N}}(s, u)$  appears as a sum of Hurwitz zeta functions

$$\zeta_{\mathcal{N}}(s, u) = a^s \sum_{\alpha=0}^{d-1} e_\alpha(au) \zeta_H(s - \alpha; (d-1)/2 + au),$$

its derivative at  $s = 0$  being

$$\zeta'_{\mathcal{N}}(0, u) = \sum_{\alpha=0}^{d-1} \left( \zeta'_H(-\alpha; (d-1)/2 + au) - \ln a \frac{B_{\alpha+1}((d-1)/2 + au)}{\alpha + 1} \right), \quad (11.3)$$

where the  $B_n(x)$  are ordinary Bernoulli polynomials.

Thus, in this case also, the only contribution not readily available for arbitrary radius  $a$  is the  $\zeta_{\mathcal{N}+1}$  one. As in Dirichlet conditions, the ball case,  $a = 1$ , is easily extracted. Again, the structure of the result is completely clear from equation (11.2) and the special cases of references [8, 21, 20] are very quickly reproduced.

## 12 Conclusion.

Our basic results are embodied in eqs. (4.8), (9.8) and (11.2). The general form of the determinants agrees with the more special expressions announced in [22].

The techniques described here have certain technical and aesthetic advantages. For example, Levitin determined the heat-kernel coefficients on the  $D$ -ball in terms of polynomials in  $d$  by fitting their general forms using values calculated for specific dimensions. Choosing  $\mathcal{N}$  to be the unit  $(D-1)$ -sphere in the preceding formalism has allowed us to find these polynomial expressions directly and much more quickly. It takes two minutes using Mathematica to evaluate the first ten polynomials. A similarly rapid computation holds for the determinants.

The method is clearly capable of being applied to other situations. One may wish to change the base  $\mathcal{N}$  or to choose a different field for physical or for mathematical reasons.

A generalisation of a slightly different character would be to replace the metric (2.1) by  $ds^2 = dr^2 + f(r^2)d\Sigma^2$  when one would be obliged to analyse the asymptotic behaviour of the new radial eigensolutions. A particularly important example is the spherical suspension,  $ds^2 = d\theta^2 + \sin^2 \theta d\Sigma^2$ ,  $0 \leq \theta \leq \theta_0$ . The asymptotic properties of the resulting Legendre functions derived by Thorne [37] have already been employed by Barvinsky *et al* [6] in a calculation of a one-loop effective action in quantum cosmology.

We reserve for later, expositions of some of these extensions.

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## A Heat-kernel polynomials for the monopole with Dirichlet boundary conditions

In this appendix we list the leading heat-kernel coefficients for the monopole with Dirichlet boundary conditions. From (5.12),

$$\begin{aligned}
\frac{(4\pi)^{d/2}}{a^d |S^d|} A_{1/2}^{\mathcal{M}} &= -\frac{1}{4} \\
\frac{(4\pi)^{D/2}}{a^d |S^d|} A_1^{\mathcal{M}} &= \frac{5d-3}{12} - \frac{d-3}{12a^2} \\
\frac{(4\pi)^{d/2}}{a^d |S^d|} A_{3/2}^{\mathcal{M}} &= \frac{(2-d)(5d-4)}{128} + \frac{(d-1)(d-3)}{48a^2} \\
\frac{(4\pi)^{D/2}}{a^d |S^d|} A_2^{\mathcal{M}} &= \frac{-7875 + 14447d - 7293d^2 + 1105d^3}{30240} + \frac{(d-1)(d-5)(5d-3)}{1440a^4} \\
&\quad - \frac{(d-1)(d-3)(5d-13)}{144a^2} \\
\frac{(4\pi)^{d/2}}{a^d |S^d|} A_{5/2}^{\mathcal{M}} &= \frac{-4992 + 9552d - 5692d^2 + 1356d^3 - 113d^4}{49152} \\
&\quad - \frac{(d-1)(d-3)(d-5)(5d-3)}{5760a^4} \\
&\quad + \frac{(d-1)(d-3)(d-4)(5d-14)}{1536a^2} \\
\frac{(4\pi)^{D/2}}{a^d |S^d|} A_3^{\mathcal{M}} &= \frac{-28999971 + 57597489d - 38150066d^2}{51891840} \\
&\quad + \frac{11356742d^3 - 1573675d^4 + 82825d^5}{51891840} \\
&\quad - \frac{(d-1)(d-3)(d-7)(35d^2 - 28d + 9)}{362880a^6} \\
&\quad + \frac{(d-1)(d-3)(d-5)(5d-3)(5d-23)}{17280a^4} \\
&\quad - \frac{(d-1)(d-3)(1105d^3 - 13923d^2 + 568789d - 74781)}{362880a^2}.
\end{aligned}$$

## B Heat-kernel polynomials for the monopole with Robin boundary conditions

The following is the list for Robin boundary conditions:

$$\frac{(4\pi)^{d/2}}{a^d |S^d|} A_{1/2}^{\mathcal{M}} = \frac{1}{4}$$

$$\begin{aligned}
\frac{(4\pi)^{D/2}}{a^d |S^d|} A_1^{\mathcal{M}} &= \frac{-3 + 5d + 24\beta}{12} + \frac{3-d}{12a^2} \\
\frac{(4\pi)^{d/2}}{a^d |S^d|} A_{3/2}^{\mathcal{M}} &= \frac{8 - 10d + 7d^2 + 32d\beta + 64\beta^2}{128} - \frac{(d-1)(d-3)}{48a^2} \\
\frac{(4\pi)^{D/2}}{a^d |S^d|} A_2^{\mathcal{M}} &= \frac{1035 - 871d - 75d^2 + 295d^3 + 2160\beta - 2304d\beta}{4320} \\
&\quad + \frac{2448d^2\beta + 5760d\beta^2 + 5760\beta^3}{4320} \\
&\quad + \frac{(d-1)(d-5)(5d-3)}{1440a^4} \\
&\quad - \frac{(d-1)(d-3)(11+5d+24\beta)}{144a^2} \\
\frac{(4\pi)^{d/2}}{a^d |S^d|} A_{5/2}^{\mathcal{M}} &= \frac{24960 - 31344d + 11668d^2 - 2836d^3 + 1587d^4 + 30720\beta}{245760} \\
&\quad + \frac{-19200d\beta - 3520d^2\beta + 14560d^3\beta + 30720\beta^2 - 25600d\beta^2}{245760} \\
&\quad + \frac{56320d^2\beta^2 + 92160d\beta^3 + 61440\beta^4}{245760} \\
&\quad + \frac{(d-1)(d-3)(d-5)(5d-3)}{5760a^4} \\
&\quad + \frac{(d-1)(d-3)(-56+6d-7d^2-64\beta-32d\beta-64\beta^2)}{1536a^2} \\
\frac{(4\pi)^{D/2}}{a^d |S^d|} A_3^{\mathcal{M}} &= \frac{1087}{1920} + \frac{1744109d^5}{259459200} + \frac{9\beta}{16} + \frac{\beta^2}{3} + \frac{\beta^3}{3} + \frac{8\beta^5}{15} \\
&\quad + d^4 \frac{(-190555 + 4176744\beta)}{51891840} \\
&\quad + d^3 \left( -\frac{1423133}{25945920} - \frac{2\beta}{21} + \frac{349\beta^2}{945} \right) \\
&\quad + d^2 \left( \frac{1300721}{3706560} \frac{293\beta}{1080} - \frac{7\beta^2}{45} + \frac{31\beta^3}{35} \right) \\
&\quad + d \left( -\frac{23787571}{28828800} - \frac{194\beta}{315} + \frac{11\beta^2}{945} - \frac{8\beta^3}{35} + \frac{16\beta^4}{15} \right) \\
&\quad - \frac{(d-1)(d-3)(d-7)(35d^2 - 28d + 9)}{362880a^6} \\
&\quad + \frac{(d-1)(d-3)(d-5)(5d-3)(25+5d+24\beta)}{17280a^4} \\
&\quad + \frac{(d-1)(d-3)}{51840a^2} \left( -10917 + 3367d - 603d^2 - 295d^3 - 10800\beta + 576d\beta \right. \\
&\quad \left. - 2448d^2\beta - 5760\beta^2 - 5760d\beta^2 - 5760\beta^3 \right).
\end{aligned}$$

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